STATE OF STRESS IN UNCONSTRAINED SHELLS OF ZERO CURVATURE

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The state of stress in a thin elastic shell of zero curvature with free edges is investigated. It is deduced that the conditions of zero moment (membrane state) formulated by Vekua for shells of positive curvature remain valid for shells of zero curvature, provided that the edges of the latter shells are not asymptotic. It is shown that the state of stress and deformation of the shell increase sharply with relatively small deviations from the conditions of zero moment.

1. We refer the neutral surface of the shell of zero curvature to lines of curvature. We denote the parameters of this system by (α, β) and assume, following [1], that α denotes the arc length along the rectilinear generator. The remaining notation is also adopted from [1].

We study a closed (without the edges $\beta = \text{const}$) shell of zero curvature whose both transverse edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$ are free from any supports. The shell is acted upon by a surface load (whose components are X, Y, Z) and the boundary forces lying in the tangential plane (the forces n_1, t_1 applied to the edge $\alpha = \alpha_1$ the forces n_2, t_2 to the edge $\alpha = \alpha_2$). The positive directions of the external forces, the displacements and the vectors of the moving trihedron are shown in Fig. 1.

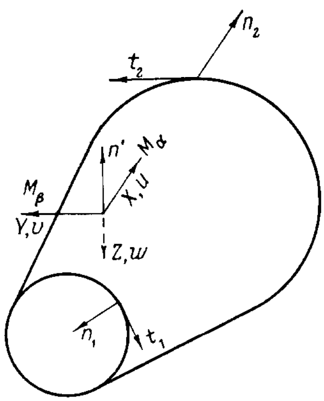

Fig. 1

2. We begin by discussing the problem of obtaining a solution of the stated problem according to the membrane theory, i.e. by integrating the membrane state equations of equilibrium with the shear boundary conditions, which in the present case have the form

$$T_1 = n_1, \quad S_1 = t_1 \quad \text{when} \quad \alpha = \alpha_1 \tag{2.1}$$
$$T_1 = -n_2, \quad S_1 = -t_2 \quad \text{when} \quad \alpha = \alpha_2$$

taken into account. This means that a membrane state of stress corresponding to the given surface load X, Y, Z and the tangential forces n_1, t_1, n_2, t_2 must be constructed. Vekua investigated in [2] such a problem for shells possessing positive Gaussian curvature throughout, and showed that the problem has a solution if and only if no work is done by the surface and the boundary loads on the displacements associated with any infinitesimal flexure which the neutral surface may undergo, under the condition that the displacements of its edge (or edges) may not be restricted in any manner. In the present paper we investigate the validity of the Vekua theorem as applied to shells of zero curvature.

First it must be noted that the Vekua theorem is invalid for shells of zero curvature with rectilinear edges. This can be shown by constructing the following example.

Example. We consider an open shell of zero curvature whose edges $\alpha = \alpha_1$, $\alpha=\alpha_2$, $\beta = \beta_1$, $\beta = \beta_2$ are free. Assume that the surface forces are absent $(X = Y = Z \equiv 0)$, that the curvilinear edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$ are load-free and, that the state of stress is induced only by the tangential forces applied to the rectilinear edges of the shell (Fig. 2).

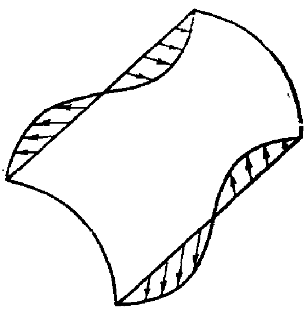

Fig. 2

The membrane state of stress fulfilling the above requirements satisfies the following membrane state equations of equilibrium of an arbitrary shell of zero curvature:

$$\frac{\partial}{\partial\alpha}(BT_1)+\frac{\partial S}{\partial\beta}-\frac{\partial B}{\partial\alpha}T_2+BX=0$$

$$\frac{\partial}{\partial\alpha}(BS)+\frac{\partial T_2}{\partial\beta}+\frac{\partial B}{\partial\alpha}S+BY=0 \qquad (2.2)$$

$$\frac{T_2}{R}+Z=0, \qquad S_1=-S_2=S$$

here B is the second coefficient of the first quadratic form and R is the principal radius of curvature (different from infinity).

The general solution of (2.2) can be written as

$$T_1=-\frac{1}{B}\int_{\alpha_1}^{\alpha}\frac{\partial}{\partial\beta}\left(\frac{f_1}{B^2}\right)d\alpha+\frac{f_2}{B}-\frac{1}{B}\int_{\alpha_1}^{\alpha}\frac{\partial}{\partial\beta}\left\{\frac{1}{B^2}\int_{\alpha_1}^{\alpha}\left[B\frac{\partial}{\partial\beta}(RZ)-B^2Y\right]d\alpha\right\}d\alpha-$$

$$-\frac{1}{B}\int_{\alpha_1}^{\alpha}\left(\frac{\partial B}{\partial\alpha}RZ+BX\right)d\alpha \qquad (2.3)$$

$$S=\frac{f_1}{B^2}+\frac{1}{B^2}\int_{\alpha_1}^{\alpha}\left[B\frac{\partial}{\partial\beta}(RZ)-B^2Y\right]d\alpha, \qquad T_2=-RZ$$

where f_1, f_2 are arbitrary functions of β. We see from these solutions that, if $X = Y = Z \equiv 0$ and the curvilinear edges are load-free, i.e. if T_1 and S satisfy the homogeneous (with $n_1 = n_2 = t_1 = t_2 \equiv 0$) conditions (2.1), then the equations of equilibrium have a single trivial solution $T_1 = S = T_2 \equiv 0$. Consequently no (non-zero) forces applied to the rectilinear edges of the shell of zero curvature can generate a membrane state of stress.

We now turn our attention to the displacements caused by the infinitesimal bending of the surface of zero curvature. Their defining equations are ([1], p. 126)

$$\frac{\partial u}{\partial\alpha}=0, \quad \frac{1}{B}\frac{\partial u}{\partial\beta}+B\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)=0, \quad \frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{1}{B}\frac{\partial B}{\partial\alpha}u-\frac{w}{R}=0$$

The quantities B and R have the same meaning as in (2.2), and the following formulas ([1], pp. 124-125) hold for the shell of zero curvature

$$B=b_0+\alpha b_1, \qquad R=r_0+\alpha r_1, \qquad B/R=b/r$$

here b, b_0, b_1, r, r_0, r_1 are functions of β only. Taking this into account we obtain the following general formulas for the displacements u, v, w

$$u = \varphi_1, \qquad v = P\frac{\partial \varphi_1}{\partial \beta} + B\varphi_2, \qquad w = \frac{R}{B}\left(\frac{\partial v}{\partial \beta} + \frac{\partial B}{\partial \alpha} u\right)$$

$$P = -B\int_{\alpha_1}^{\alpha} \frac{1}{B^2}\, d\alpha \tag{2.4}$$

where φ_1 and φ_2 are arbitrary functions of β.

For $\beta = \text{const}$, i.e. on the rectilinear edges, u and v vary linearly in α and the external forces can be chosen on these edges without any difficulty in shch a manner, that no work is done by them on the variables u and v. This implies that the Vekua theorem cannot be applied to a nonclosed shell of zero curvature.

3. We now consider a closed shell of zero curvature. The work L done by all external forces on its displacements is given by

$$L = \int_{\beta_1}^{\beta_2} (-un_1 - vt_1)\, B_1 d\beta + \int_{\beta_1}^{\beta_2} (un_2 + vt_2)\, B_2 d\beta +$$

$$+ \int_{\beta_1}^{\beta_2}\int_{\alpha_1}^{\alpha_2} (Xu + Yv + Zw)\, B\, d\alpha\, d\beta, \qquad B_i = B|_{\alpha=\alpha_i} \quad (i = 1, 2)$$

Let us insert into the above formula the expressions (2.4) for u, v and w and perform the integration by parts with respect to β so as to eliminate $d\varphi_1 / d\beta$ and $dv / d\beta$. Assuming that the corresponding integrands are continuous in β, we obtain

$$L = -\int_{\beta_1}^{\beta_2} t_1 B_1^2 \varphi_2\, d\beta + \int_{\beta_1}^{\beta_2} t_2 B_2^2 \varphi_2\, d\beta - \int_{\beta_1}^{\beta_2} n_1 B_1 \varphi_1\, d\beta +$$

$$+ \int_{\beta_1}^{\beta_2} n_2 B_2 \varphi_1\, d\beta + \int_{\beta_1}^{\beta_2}\int_{\alpha_1}^{\alpha_2} \left[BY - \frac{\partial}{\partial \beta}(RZ)\right] B\varphi_2\, d\alpha\, d\beta +$$

$$+ \int_{\beta_1}^{\beta_2}\int_{\alpha_1}^{\alpha_2} \left(BX + \frac{\partial B}{\partial \alpha} RZ\right)\varphi_1\, d\alpha\, d\beta - \int_{\beta_1}^{\beta_2}\int_{\alpha_1}^{\alpha_2} \varphi_1 \frac{\partial}{\partial \beta} P\left[BY -\right.$$

$$\left. - \frac{\partial}{\partial \beta}(RZ)\right] d\alpha\, d\beta - \int_{\beta_1}^{\beta_2} \varphi_1 \frac{\partial}{\partial \beta}(t_2 B_2 P_2)\, d\beta + \left\{\int_{\alpha_1}^{\alpha_2} vRZ\, d\alpha +\right.$$

$$\left. + \varphi_1 t_2 B_2 P_2 + \varphi_1 \int_{\alpha_1}^{\alpha_2} P\left[BY - \frac{\partial}{\partial \beta}(RZ)\right] d\alpha\right\}_{\beta_1}^{\beta_2} \tag{3.1}$$

$$P_2 = P|_{\alpha=\alpha_2}$$

If the obvious periodicity requirements hold on traversing the contour of the transverse cross section, then the integrated part of (3.1) contained within the braces will vanish. This we assume, requiring in addition that L vanish for any φ_1, φ_2, i.e. that no work is done by the external forces on the flexural displacements. This is obviously equivalent to the requirement that the coefficients of φ_1 and φ_2 appearing under the integrals (in β) vanish separately. Hence

$$\int_{\alpha_1}^{\alpha_2}\left[B\frac{\partial}{\partial\beta}(RZ)-B^2Y\right]d\alpha+t_1B_1^2-t_2B_2^2=0 \tag{3.2}$$

$$\frac{\partial}{\partial\beta}\int_{\alpha_1}^{\alpha_2}P\left[\frac{\partial}{\partial\beta}(RZ)-BY\right]d\alpha+\int_{\alpha_1}^{\alpha_2}\left(BX+\frac{\partial B}{\partial\alpha}RZ\right)d\alpha-$$
$$-\frac{\partial}{\partial\beta}(t_2B_2P_2)-n_1B_1+n_2B_2=0$$

4. All membrane states of stress, i.e. the solutions of (2.2), are defined by (2.3). The requirement that the first two boundary conditions (2.1) hold, yields the following formulas for the arbitrary functions f_1 and f_2

$$f_1=t_1B^2{}_1,\quad f_2=n_1B_1$$

Inserting this into (2.3) and demanding that the last two conditions of (2.1) hold, we obtain the following two necessary and sufficient conditions of existence of the desired membrane state of stress

$$\int_{\alpha_1}^{\alpha_2}\left[B\frac{\partial}{\partial\beta}(RZ)-B^2Y\right]d\alpha+t_1B_1^2-t_2B_2^2=0 \tag{4.1}$$

$$\frac{\partial}{\partial\beta}\int_{\alpha_1}^{\alpha_2}\frac{1}{B^2}\int_{\alpha_1}^{\alpha}B\left[\frac{\partial}{\partial\beta}(RZ)-BY\right]d\alpha\, d\alpha+\int_{\alpha_1}^{\alpha_2}\left(BX+\frac{\partial B}{\partial\alpha}RZ\right)d\alpha+$$
$$+\int_{\alpha_1}^{\alpha_2}\frac{\partial}{\partial\beta}\left(\frac{B_1^2}{B^2}t_1\right)d\alpha-n_1B_1+n_2B_2=0 \tag{4.2}$$

The relations (4.1), (4.2) are equivalent to (3.2). This is obvious in the case of (4.1). In the case of (4.2) we can note that by (2.4) we have

$$\frac{1}{B^2}=-\frac{\partial}{\partial\alpha}\left(\frac{P}{B}\right)$$

and eliminate the inner integral in the first term of the left-hand side using the integration by parts. The resulting equation can easily be transformed with the help of (4.1) into (3.2). This proves the validity of the Vekua theorem for a closed shell of zero curvature with the transverse edges following the lines of curvature (the only requirement being that the components of the surface load and the edge forces are sufficiently smooth functions of the points belonging to the transverse cross section of the shell).

5. We shall now consider a problem whose formulation includes moments and use the example of a circular cylindrical shell of medium length to study how the asymptotic behavior (in h) of the stress-strain state varies with the conditions of the Vekua theorem being satisfied or violated.

The position of a point on the neutral surface is defined in terms of the relative distance ξ, measured in fractions of the shell radius r along the generator and the angular coordinate θ, and we assume that the whole neutral surface is defined by the inequalities

$$0\leqslant\theta\leqslant 2\pi,\quad -1\leqslant\xi\leqslant 1$$

We assume the boundary load to be zero ($n_1=t_1=n_2=t_2=0$) and the surface load to be defined by

$$X = X_m \cos m\theta, \quad Y = Y_m \sin m\theta, \quad Z = Z_m \cos m\theta \tag{5.1}$$

where $m = 2$, stipulating at the same time that the surface load is self-balancing over the whole shell and that it does not vary much.

The transverse shell edges $\xi = -1$ and $\xi = 1$ are assumed to be free, and to simplify matters we restrict ourselves to the case when $Y_m(\xi)$ and $Z_m(\xi)$ are even functions, while $X_m(\xi)$ is an odd function of ξ. Then the problem becomes symmetrical with respect to the cross section $\xi = 0$ and only phenomena taking place within the region $0 \leqslant \xi \leqslant 1$ need be considered.

We employ the method of partitioning described in [1], i.e. we seek the general state of stress Q in the form of four terms

$$Q = Q^{(p)} + Q^{(m)} + Q^{(b)} + Q^{(e)}$$

where $Q^{(p)}$ is the state of stress corresponding to the particular integral, $Q^{(m)}$ is the membrane state of stress, $Q^{(b)}$ is the purely moment-induced state of stress and $Q^{(e)}$ is the simple edge effect (in all cases Q denotes the collection of stresses, moments, displacements and angles of rotation of the state of stress under consideration). All these states were constructed in [1] (ch. 6, 15 and 17) and the formulas defining the corresponding quantities are (*)

$$S_1^{(p)} = -r\int_1^{\xi}(mZ_m + Y_m)\,d\xi \sin m\theta$$

$$T_1^{(p)} = r\int_1^{\xi}\left[-X_m + m\int_1^{\xi}(mZ_m + Y)\,d\xi\right]d\xi \cos m\theta$$

$$T_1^{(m)} = C_1 \cos m\theta, \quad S_1^{(m)} = 0, \quad G_1^{(m)} = -C_1 r\frac{m^2}{3}\cos m\theta, \quad N_1^{(m)} = 0$$

$$T_1^{(b)} = -\eta^4 C_2 \frac{m^3(m^2-1)(m-1)}{6r(1-\sigma^2)}\xi^2 \cos m\theta$$

$$S_1^{(b)} = \eta^4 C_2 \frac{m^2(m^2-1)(m-1)}{3r(1-\sigma^2)}\xi \sin m\theta \tag{5.2}$$

$$G_1^{(b)} = \eta^4 C_2 \frac{m(m-1)}{3(1-\sigma^2)}\sigma \cos m\theta, \quad N_1^{(b)} = 0$$

$$2Ehw^{(e)} = (C_3\cos\zeta + C_4\sin\zeta)e^{\zeta}\cos m\theta$$

$$T_1^{(e)} = -\eta^2 m^2 \nu^{-2} r^{-1}(C_4\cos\zeta - C_3\sin\zeta)e^{\zeta}\cos m\theta$$

$$T_2^{(e)} = -r^{-1}(C_3\cos\zeta + C_4\sin\zeta)e^{\zeta}\cos m\theta$$

$$S_1^{(e)} = -\eta\,{}^1\!/_2\sqrt{2}\,m\nu^{-1}r^{-1}[(C_3 - C_4)\cos\zeta + (C_3 + C_4)\sin\zeta]e^{\zeta}\sin m\theta$$

$$G_1^{(e)} = -\eta^2\nu^{-2}(C_4\cos\zeta - C_3\sin\zeta)e^{\zeta}\cos m\theta$$

$$N_1^{(e)} = \eta\,{}^1\!/_2\sqrt{2}\,\nu^{-1}r^{-1}[(C_3 - C_4)\cos\zeta + (C_3 + C_4)\sin\zeta]e^{\zeta}\cos m\theta$$

$$\nu^4 = 3(1-\sigma^2), \quad \zeta = {}^1\!/_2\sqrt{2\nu}\,(\xi - 1)\,\eta$$

where η is a small parameter given by

$$\eta = \sqrt{h/R} \tag{5.3}$$

*) For pure moment-induced state of stress [1] gives only the displacements, but the forces and moments defined by the latter can easily be obtained using the methods given in [1] ch. 17.

Since the unknown quantities (functions of θ) vary as $\sin m\theta$ or $\cos m\theta$, constants replace the arbitrary functions in the above formulas. C_1 and C_2 are the constants of the membrane state of stress and of the purely moment-induced state of stress (since the problem is symmetrical, each state retains one constant), while C_3 and C_4 are the constants associated with the simple edge effect at the edge $\xi = 1$.

When a load of the form (5.1) is applied to a circular cylindrical shell, the conditions of the Vekua theorem become

$$\int_{-1}^{1} (mZ_m + Y_m)\, d\xi = 0, \qquad \int_{-1}^{1} [X_m + \xi m (mZ_m + Y_m)]\, d\xi = 0 \tag{5.4}$$

In the present case these conditions are equivalent to

$$S_1^{(p)}\big|_{\xi=1} = T_1^{(p)}\big|_{\xi=1} = 0 \tag{5.5}$$

The possibility that (5.5) may not hold in the principal or second order terms cannot, however, be dismissed; therefore we assume that

$$T_1^{(p)}\big|_{\xi=1} = \eta^{\mu} \sum \eta^{s} T_{1(s)}^{(p)}, \qquad S_1^{(p)}\big|_{\xi=1} = \eta^{\mu} \sum \eta^{s} S_{1(s)}^{(p)} \tag{5.6}$$

here μ is an integer determining the accuracy with which (5.5) are expected to hold. It is assumed that (5.5) hold to within the order of

$$\varepsilon = O(\eta^{\mu})$$

At the edge $\xi = 1$ the condition that forces and moments are absent must hold. This leads to the following system of algebraic equations defining the constants C_1, C_2, C_3, C_4:

$$\begin{gathered} T_1^{(p)} + \eta^{-a} C_1^* - \eta^{-b+2} 4r^{-1} C_2^* - \eta^{-c+2} 4\nu^{-2} r^{-1} C_4^* = 0 \\ S_1^{(p)} + \eta^{-b+2} C_2^* - \eta^{-c+1} \sqrt{2}\, \nu^{-1} r^{-1} (C_3^* - C_4^*) = 0 \\ \eta^4 G_{1*} - \eta^{-a+4}\, {}^4/_3\, r C_1^* + \eta^{-b+2}\, {}^2/_9\, C_2^* - \eta^{-c+2} \nu^{-2} C_4^* = 0 \\ \eta^4 N_{1*} + \eta^{-c+1}\, {}^1/_2 \sqrt{2}\, \nu^{-1} r^{-1} (C_3^* - C_4^*) = 0 \end{gathered} \tag{5.7}$$

where

$$\begin{gathered} C_1 = \eta^{-a} C_1^*, \quad C_2 = \eta^{-b-2} C_2^*, \quad C_3 = \eta^{-c} C_3^* \\ C_4 = \eta^{-c} C_4^*, \quad G_1^{(p)} = \eta^4 G_{1*}, \quad N_1^{(p)} = \eta^4 N_{1*} \end{gathered} \tag{5.8}$$

We neglect, for simplicity, the correction terms due to the torsional moments appearing in S_1 and N_1. The expressions for G_{1*} and N_{1*} appearing in the last two equations of (5.7) could easily be written out in full, but they are complicated and shall not be required in what follows. We only note that the powers of η in the expressions for $N_1^{(p)}$ and $G_1^{(p)}$ appearing in (5.8) are chosen so, that

$$G_{1*} = O(T_1^{(p)}), \qquad N_{1*} = O(T_1^{(p)}) \tag{5.9}$$

We assume that the power indices a, b, c appearing in (5.8) are integers and that the constants C_i^* are of the order $O(\eta^0)$. Then (5.8) is used to find the asymptotics of constants C_i, and by virtue of (5.2), the required asymptotics of the state of stress of the shell is defined.

The indices a, b, c in (5.7) must be chosen so, that the passage $\eta \to 0$ yields an iterative process of solution of (5.7), in which the initial approximation system is solvable and independent of η. This is achieved by choosing

$$a = -\mu, \quad b = 2 - \mu, \quad c = 2 - \mu \tag{5.10}$$

To prove this we write the required constants in the form

$$C_i^* = \sum_{s=0} \eta^s C_{i(s)}, \qquad i = 1, 2, 3, 4 \tag{5.11}$$

insert (5.11) into (5.7), take (5.6) into account, and equate the coefficients of like powers of η to zero in each equation, to obtain

$$\begin{aligned} &T_{1\,(s)}^{(p)} + C_{1(s)} - 4r^{-1}C_{2(s)} - 4\nu^{-2}r^{-1}C_{4(s)} = 0 \\ &S_{1(s)}^{(p)} + 4r^{-1}C_{2(s)} - \sqrt{2}\,\nu^{-1}r^{-1}(C_{3(s+1)} - C_{4(s+1)}) = 0 \\ &G_{1(s-4+\mu)}^{(p)} - {}^4/_3\, rC_{1(s-4)} + {}^2/_9\, C_{2(s)} - \nu^{-2}C_{4(s)} = 0 \\ &N_{1(s-4+\mu)}^{(p)} + {}^1/_2\sqrt{2}\,\nu^{-1}r^{-1}(C_{3(s+1)} - C_{4(s+1)}) = 0\,. \end{aligned} \tag{5.12}$$

We assume that

$$G_{1*} = \sum \eta^s G_{1(s)}^{(p)}, \qquad N_{1*} = \sum \eta^s N_{1(s)}^{(p)}$$

The sequence of systems of algebraic equations (5.12) in which the quantities with negative indices are assumed zero, yields the following recurrent formulas defining the groups of constants

$$C_{i(s)} \qquad (i = 1, 2, 3, 4) \tag{5.13}$$

in the order of increasing (s).

Let us suppose that the constants (5.13) have been found for $s = 0, 1, \ldots k - 1$. Assuming that the quantities accompanied by the superscript (p) are also known, we can obtain (5.13) for $s = k$ by putting $s = k$ in (5.12) and adding to the resulting equations the fourth equation of (5.12) after substituting $s = k - 1$ into it. This yields a system of five equations with five unknowns C_{ik} $(i = 1, 2, 3, 4)$ and $C_{3(k+1)} - C_{4(k+1)}$ which, as can easily be checked, always has a solution.

It can easily be shown that (5.13) can also be obtained for $s = 0$. Putting $s = 0$ in (5.12) and adding the fourth equation of (5.12) in which we put $s = -1$, we obtain

$$\begin{aligned} C_{1(0)} &= -T_{1(0)}^{(p)} - {}^{11}/_9\, S_{1(0)}^{(p)} + 4r^{-1}G_{1(\mu-4)}^{(p)} - {}^{22}/_9\, N_{1(\mu-4)}^{(p)} \\ C_{2(0)} &= -{}^1/_4\, r\,(S_{1(0)}^{(p)} + 2N_{1\,(\mu-4)}^{(p)}) \\ C_{3(0)} = C_{4(0)} &= -{}^1/_{18}\,\nu^2 r S_{1(0)}^{(p)} + \nu^2 G_{1(\mu-4)}^{(0)} - {}^1/_9\,\nu^2 r N_{1(\mu-4)}^{(p)} \end{aligned} \tag{5.14}$$

which proves the proposition.

6. We consider the asymptotics of the stress-strain state of the shell, arising from (5.9). Formulas connecting the forces T and moments G with the stresses can be written as

$$\sigma_T = {}^1/_2 T\,/\,h, \qquad \sigma_G = {}^3/_2 G\,/\,h^2 \tag{6.1}$$

where T denotes any force and G any moment.

This, together with the formulas (5.2), (5.3), (5.8) and (5.9), yields the following stress asymptotics

$$\begin{aligned} \sigma_T^{(p)} &= \eta^{-2} O(T^{(p)}), & \sigma_G^{(p)} &= \eta^{\circ} O(T^{(p)}) \\ \sigma_T^{(m)} &= \eta^{\mu-2} O(T^{(p)}), & \sigma_G^{(m)} &= \eta^{\mu} O(T^{(p)}) \\ \sigma_T^{(e)} &= \eta^{\mu-4} O(T^{(p)}), & \sigma_G^{(e)} &= \eta^{\mu-4} O(T^{(p)}) \\ \sigma_T^{(b)} &= \eta^{\mu-2} O(T^{(p)}), & \sigma_G^{(b)} &= \eta^{\mu-4} O(T^{(p)}) \end{aligned} \tag{6.2}$$

while the elasticity relations

$$2Eh\frac{\partial u}{\partial \alpha} = T_1 - \sigma T_2, \quad 2Eh\left[\frac{1}{B}\frac{\partial u}{\partial \beta} + B\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right] = T_2 - \sigma T_1$$

$$2Eh\left(\frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{B}\frac{\partial B}{\partial \alpha}u - \frac{w}{R}\right) = 2(1+\sigma)S \qquad (6.3)$$

yield the displacement asymptotics for the particular integral and for the membrane state of stress.

An assumption that the displacements do not vary very much, i. e. that differentiation does not affect their asymptotic order, enables us to write

$$2EhU^{(p)} = O(T^{(p)}), \quad 2EhU^{(m)} = O(T^{(m)}) = \eta^{\mu}\, O(T^{(p)}) \qquad (6.4)$$

where U is any one displacement.

The latter estimates are not suitable for the purely moment-induced state of stress where the principal parts of the left-hand sides of (6. 3) vanish, nor for the edge effect for which the assumption that no large variations in the displacements take place does not hold. For the purely moment-induced state of stress, relations of the form

$$2Eh\varkappa_1 = 3h^{-2}(G_1 - \sigma G_2), \; 2Eh\varkappa_2 = 3h^{-2}(G_2 - \sigma G_1)$$

$$2Eh\tau = 3h^{-2}(1+\sigma)H_1$$

must be used instead of (6. 3), and these relations yield, on expressing $\varkappa_1$, $\varkappa_2$, τ in terms of displacements

$$2EhU^{(b)} = \eta^{-4}O(G^{(b)}) = \eta^{\mu-4}O(T^{(p)}) \qquad (6.5)$$

The edge effect displacement asymptotics follows from (5. 2), (5. 3) and (5. 8) and can be expressed by

$$2EhU^{(e)} = O(T_2^{(e)}) = \eta^{\mu-2}O(T^{(p)}) \qquad (6.6)$$

where $U^{(e)}$ denotes the largest displacement due to the edge effect, i. e. the displacement $w^{(e)}$.

In the following the quantity Q will be understood to exceed P by r orders of magnitude if $P = QO(\eta^r)$ (if the relative thickness of the shell is 0. 01, then the statement: "increased by an order of magnitude" will imply the appearance of an additional multiplier of the order of 10).

Let us put $\mu = 4$ in (6. 2), (6. 4), (6. 5) and (6. 6), i. e. let us assume that the conditions of the Vekua theorem hold with the accuracy of up to the order of

$$\varepsilon = O(\eta^4) = O(h^2)$$

Then the stresses associated with the particular integral will exceed those due to the purely moment-induced state of stress and the edge effect by two orders of magnitude, i. e. the methods of the membrane theory which can be used in the present case, give a stress pattern which is formally correct. When the moment theory is used, the only significant corrections appear in the displacements since $U^{(b)}$, as well as $U^{(p)}$ and $U^{(m)}$, are all of the same order. (This follows naturally since the membrane theory does not, in general, allow a unique determination of the displacements in an unconstrained shell).

Formally, when an unconstrained shell is loaded in accordance with the conditions of the Vekua theorem, its behavior resembles that of a supported shell (it can be shown that in the latter shell the asymptotic stresses and the deformability are of the same order). This fact, however, is of no practical value, since the stress-strain asymptotics in an unconstrained shell is unstable (small deviations in the load produce significant changes in the stresses or displacements). For example, when $\mu = 1$, i.e. when the conditions of zero moment are violated in the quantities of the order of $O(\eta) = O(h^{1/2})$ (a 10% error with the relative shell thickness of 0.01), the stresses increase by an order of magnitude (roughly speaking by 10 times) and the displacements by three orders of magnitude (1000 times).

For $\mu = 0$ and the conditions of zero moment are violated in the principal terms, the stresses increase by two orders of magnitude (obviously we assume that the linear theory of shells still holds, i.e. that the loads are much smaller than the ones that the shell is usually called to carry).

7. The results of the asymptotic analysis were verified by solving the problem numerically, according to the plane theory of shells with the moments taken into account. The algorithm for such a solution is discussed in detail in [1], ch. 10.

The load was given in the form

$$X = \sin n\xi \cos 2\theta, \quad Y = \cos n\xi \sin 2\theta, \quad Z = \cos n\xi \cos 2\theta \tag{7.1}$$

The parameters chosen for use in the computations were

$$\eta^2 = h / r = 0.01, \quad \sigma = 1/3, \quad 2Eh / (1 - \sigma^2) = 1, \quad r = 1$$

Under the load (7.1) the second conditions of (5.4) is satisfied identically, while the first one is reduced to

$$\sin n = 0 \tag{7.2}$$

and holds, if $n = \pi k$ $(k = 1, 2, 3, \ldots)$.

In the course of computations we have set $k = 1$. We have also considered the cases when (7.2) is changed by 0.1^{μ} $(\mu = 3, 2, 1, 0)$. This was done by assigning the values $(0.999\pi, 0.99\pi, 0.9\pi$ and $0.5\pi)$ to n and performing the computations for each of these values. For $n = \pi$, the zero moment conditions are satisfied exactly, but the membrane theory itself has an intrinsic error which is at least of the order $O(h^2) = O(\eta^4)$ (see [1], ch. 15) and the value of $\mu = 4$ was set to correspond to $n = \pi$.

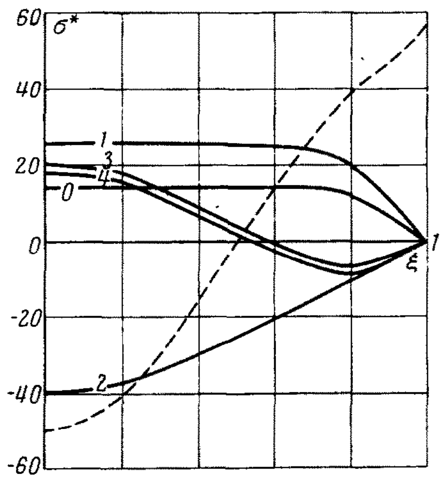

Fig. 3

The results of the computations are given graphically.

The graphs of σ^* versus ξ are given in Fig. 3 as solid lines, each accompanied by a number indicating the corresponding value of μ. The quantity σ^* is connected with σ_G, the latter denoting the stress induced by the moment G_1 by the following formulas

$$\sigma_G = \sigma^* \cos 2\theta \quad (\mu = 2, 3, 4) \qquad \sigma_G = 10^{2-\mu} \sigma^* \cos 2\theta \quad (\mu = 0, 1)$$

in which the choice of the scale factor is made in full agreement with the theoretical asymptotics given by (6.2). The validity of the latter is confirmed by the fact that σ^* is of the same order in all versions (the ratio of the largest ordinate values does not exceed three).

The broken line in Fig. 3 depicts the amplitude values of the stresses σ_T versus the maximum tangential force T_2, for $\mu = 4$. Comparing σ_T with σ_G we see, that for $\mu = 4$, $\sigma_T > \sigma_G$. This means that when the conditions of zero moment hold, the state of stress can indeed be called moment-free, although this is not a strict definition. Comparing, on the other hand, σ_T with σ_G when $\mu = 2$ we see, that their maximum ordinates are nearly the same. This also corresponds to the asymptotics (6.2), the latter implying that the stresses induced by the forces and moments become commensurable, when $\mu = 2$.

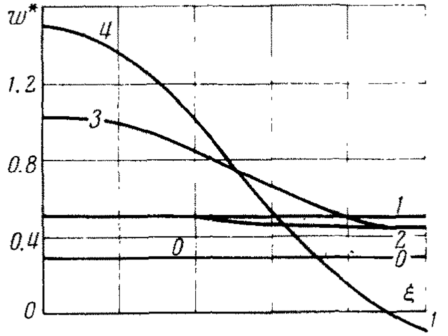

Fig. 4

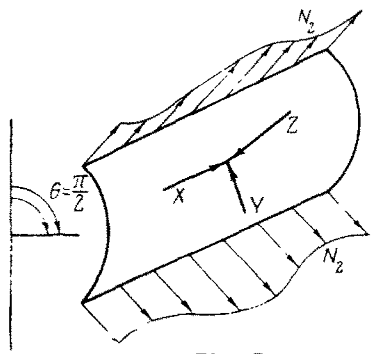

Fig. 5

Fig. 4 depicts the auxilliary function w^* related to the deflection w as follows:

$$w = 10^{4-\mu} w^* \cos 2\theta \quad (\mu = 0,\ 1,\ 2,\ 3), \qquad w = 4 \cdot 10^{4-\mu} w^* \cos 2\theta \quad (\mu = 4)$$

The scale factor here is chosen in accordance whith the theoretical asymptotics (6.4)–(6.6), except for the case $\mu = 4$ for which an additional multiplier is introduced. The graphs confirm the theory, although the case of $\mu = 4$ deviates from it somewhat. This is however to be expected, since (6.4)–(6.6) imply that for $\mu < 4$, w (at some distance from the edge) is basically defined by the purely moment-induced state of stress, and by the sum of the moment-induced and membrane states of stress when $\mu = 4$.

8. The jump in stresses and consequently in the deformation which occurs when (5.4) are violated, is inevitable. It follows from the static concepts and must not be attributed to the shortcomings of the theory of shells.

Let us cut, from the circular cylindrical shell acted upon by the load (7.1), a segment (Fig. 5) defined by the inequalities

$$-1 \leqslant \xi \leqslant 1, \qquad {}^1/_4\,\pi \leqslant \theta \leqslant {}^3/_4\,\pi$$

The transverse shell edges are free, therefore the surface load applied to the cut-out part should be balanced by the forces and moments applied to the edges $\theta = {}^1/_4\,\pi$ and $\theta = {}^3/_4\,\pi$.

When the problem is symmetrical with respect to the cross section $\xi = 0$, these forces and moments have the form ([1], pp. 220–222)

$$T_2 = T_{22} \cos 2\theta,\ G_2 = G_{22} \cos 2\theta, \qquad S_2' = S_{22}' \sin 2\theta,\ N_2' = N_{22}' \sin 2\theta \tag{8.1}$$

consequently on the rectilinear edges we have $T_2 = G_2 = 0$.

Let us project the forces acting on the cut-out segment on the arc $\theta = {}^1/_2\,\pi$ perpendicular to the shell axis

$$\int_{-1}^{1} \int_{{}^1/_4\pi}^{{}^3/_4\pi} [Z \cos(\theta - {}^1/_2\pi) + Y \sin(\theta - {}^1/_2\pi)]\, r\, d\theta\, d\xi = \sqrt{2} \int_{-1}^{1} N_2'\, r\, d\theta\, d\xi$$

Inserting (7.1) into the above expression we obtain

$$2\sin n = -\int_{-1}^{1} N_2' \, d\xi \tag{8.2}$$

This, together with the assumption that the functions appearing in the above expressions do not undergo large changes, yields

$$N_2' = O(\sin n)$$

But according to our assumption $\sin n$ represents the error of the zero-moment conditions, hence

$$N_2' = O(\eta^{\mu}) \tag{8.3}$$

The fourth equation of equilibrium

$$\frac{\partial H_1}{\partial \xi} - \frac{\partial G_2}{\partial \theta} + rN_2 = 0$$

together with (8.3) and (6.1), yields the following estimate:

$$\sigma_G = O(\eta^{\mu-4})$$

which is in full agreement with (6.2).

BIBLIOGRAPHY

1. Gol'denveizer, A. L., Theory of Elastic Thin Shells. M., Gostekhizdat, 1953.
2. Vekua, I. N., Generalized Analytic Functions. M., Fizmatgiz, 1959.

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ON BIFURCATION OF THE EQUILIBRIUM MODES OF AN ELASTOPLASTIC ROD AND ANNULUS UNDER PROLONGED LOADING CONDITIONS

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Stability problems of a rectilinear rod and a circular annulus under compression beyond the elastic limit are examined on the basis of the prolonged loading concept.

For an idealized elastoplastic rod model Shanley [1] showed that the least critical value of the axial compressive force is realized under the condition of continuous growth of the external loading during buckling. This result was obtained by static methods and was later expanded by a number of authors [2-5].

Proceeding from the assumption of equilibrium of the deformation process beyond the elastic limit, the stability of a compressed rod is examined taking into account the actual position of the boundary separating the elastic and plastic domains during buckling. By an asymptotic solution of the nonlinear elastoplastic equilibrium equations the character of the branching of the equilibrium modes in the neighborhood of the bifurcation point is investigated.

The bending equations in the post-critical state are obtained by a variational method and generalize the Euler elastic equation to the case of elastoplastic deformations. In